



11.7 Normal operators

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Introduction

- In this topic, we will
 - Define what it is for a linear operator to be normal
 - Investigate how such linear operators and their adjoints affect the length of vectors
 - See that the null space and range are identical for both a normal linear operator and its adjoint





Definition

- A linear operator $A: \mathcal{U} \rightarrow \mathcal{U}$ is said to be *normal* if it commutes with its adjoint

$$AA^* = A^*A$$





Non-examples

- This is not normally true
 - For example,

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$$

$$AA^T = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 13 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 10 \end{pmatrix}$$





Examples of normal operators

Theorem

Every self-adjoint and skew-adjoint linear operator is normal.

Proof:

If $A = A^*$, then $AA^* = A^2 = A^*A$.

Thus, a self-adjoint operator is normal.

If $A^* = -A$, then $AA^* = -A^2 = A^*A$.

Thus, a skew-adjoint operator is normal. ■





Examples of normal operators

- As a consequence:
 - Every real symmetric matrix is normal
 - Every real skew-symmetric matrix is normal
 - Every conjugate symmetric matrix is normal
 - Every conjugate skew-symmetric matrix is normal
- There are other matrices that are neither symmetric nor skew-symmetric that are normal

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = A + I_3$$





Consequences

Theorem

If A is normal, then $\|A\mathbf{u}\|_2 = \|A^*\mathbf{u}\|_2$ for all $\mathbf{u}$.

Proof:

$$\begin{aligned}\|A\mathbf{u}\|_2^2 &= \langle A\mathbf{u}, A\mathbf{u} \rangle \\ &= \langle \mathbf{u}, A^* A\mathbf{u} \rangle \\ &= \langle \mathbf{u}, AA^*\mathbf{u} \rangle \\ &= \langle A^*\mathbf{u}, A^*\mathbf{u} \rangle \\ &= \|A^*\mathbf{u}\|_2^2\end{aligned}$$

Thus, $\|A\mathbf{u}\|_2 = \|A^*\mathbf{u}\|_2$. ■





Consequences

- Note that this is not true in general:
 - For example,

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$\left\| \begin{pmatrix} 1 \\ 5 \end{pmatrix} \right\|_2 = \sqrt{26}$$

$$\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\left\| \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\|_2 = \sqrt{20}$$





Equality of null space and range

Theorem

If A is a normal linear operator,
then $\text{null}(A) = \text{null}(A^*)$ and $\text{range}(A) = \text{range}(A^*)$.

Proof:

If $A\mathbf{u} = \mathbf{0}$, then $\|A\mathbf{u}\|_2 = 0$.

From our last theorem, it follows that $\|A^*\mathbf{u}\|_2 = 0$

Thus, $A^*\mathbf{u} = \mathbf{0}$.

Therefore, the null spaces are equal.





Equality of null space and range

Theorem

If A is a normal linear operator,
then $\text{null}(A) = \text{null}(A^*)$ and $\text{range}(A) = \text{range}(A^*)$.

Sketch of the proof:

If $\mathbf{u} \in \text{null}(A) = \text{null}(A^*)$, then $A\mathbf{u} = A^*\mathbf{u} = \mathbf{0}$

Thus, $\langle A\mathbf{u}, \mathbf{v} \rangle = \langle A^*\mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{v}$.

Thus, $\langle \mathbf{u}, A^*\mathbf{v} \rangle = \langle \mathbf{u}, A^{**}\mathbf{v} \rangle = \langle \mathbf{u}, A\mathbf{v} \rangle = 0$ for all $\mathbf{v}$.

Thus, each vector in the null space is perpendicular to all vectors each of the ranges.

But $\dim(\text{null}(A)) + \dim(\text{range}(A)) = n$ and the same for A^* ,
thus, these two ranges must be equal. ■





Equality of null space and range

- Let's look at this normal matrix:

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$(A \mid \mathbf{0}) \sim \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$(A^T \mid \mathbf{0}) \sim \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$





Equality of null space and range

- This matrix is not normal:

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$(A \mid \mathbf{0}) \sim \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$(A^T \mid \mathbf{0}) \sim \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$





Summary

- Following this topic, you now
 - Understand what it means for a linear operator to be normal:
$$AA^* = A^*A$$
 - Understand that all self-adjoint and skew-adjoint linear operators are normal, but there are others
 - Know that for normal matrices, $\|A\mathbf{u}\|_2 = \|A^*\mathbf{u}\|_2$
 - Know that if a linear operator is normal,
then the null space and the range of both A and A^* are identical





References

- [1] https://en.wikipedia.org/wiki/Normal_matrix
- [2] https://en.wikipedia.org/wiki/Normal_operator





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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